

NOISE REDUCTION FOR SENSOR COUNTING PROBLEM USING DISCRETE EULER CALCULUS

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ABSTRACT. This paper proposes a method to reduce noise in acyclic sensor networks enumerating targets using the integral theory with respect to Euler characteristic. For an acyclic network (a partially ordered set) equipped with sensors detecting targets, we find reducible points for enumerating targets, as a generalization of weak beat points (homotopically reducible points). This theory is useful for improving the reliability and optimization of acyclic sensor networks.

1. INTRODUCTION

The original ideal that was used to apply the integration theory with respect to Euler characteristic (known as *Euler calculus* or *Euler integration*) to sensor networks was developed by Baryshnikov and Ghrist [BG09]. They proposed a method for enumerating targets lying on a sensor field using Euler calculus.

The author drew inspiration from their work and established a discrete version of their work for finite categories, especially for finite partially ordered set (posets for short) in [Tan16]. By regarding an acyclic network flowing only in one direction as a finite poset, the discrete version of Euler calculus enumerates targets lying on an acyclic network with sensors detecting the targets.

Fact 1.1 (Theorem 4.1 of [Tan16]). *Let (P, T, h) be an acyclic sensor network consisting of a finite poset P with sensors, a set T of targets lying on P , and the counting function h on P given by the sensors detecting the targets. The number of targets $T^\#$ is equal to the Euler calculus of the counting function:*

$$\int_P h d\chi = T^\#.$$

This paper is a continuation of the study of [Tan16] from the viewpoint of the reliability or optimization of our sensor network theory. The research question pertains to a practical problem: when some sensors break down and return an incorrect counting function, can we enumerate the correct number of targets by using Euler calculus?

We show that a homotopically reducible point (weak down-beat point) does not affect the Euler calculus. This is a naturally prospective result because the Euler characteristic is a homotopical invariant. Furthermore, we introduce a more generalized notion termed χ -points, and show the same property as above. We call a point x in a finite poset P to be a χ -point if $P_{>x} = \{y \in P \mid y > x\}$ has the Euler characteristic 1.

Main Theorem 1 (Theorem 4.1). Let x be a χ -point of a finite poset P . If two functions h, h' on P satisfy $h(y) = h'(y)$ for any $y \neq x$, then

$$\int_P h d\chi = \int_P h' d\chi.$$

The theorem above states that we do not need sensors at weak down-beat points to enumerate targets lying on an acyclic network. This is a useful result for cost-cutting or maintenance of sensors. Moreover, we can simplify the computation of the Euler calculus by reducing χ -points and using the restricted counting function.

Main Theorem 2 (Theorem 4.11). Let x be a χ -point of a finite poset P . For a function h on P ,

$$\int_P h d\chi = \int_{P \setminus \{x\}} (h|_{P \setminus \{x\}}) d\chi.$$

The remainder of this paper is organized as follows. Section 2 recalls the theory of discrete Euler calculus for functions on posets based on [Tan16]. Section 3 describes the homotopy theory for finite posets, including the definition of (weak) beat points. In Section 4, we discuss our main theorems with respect to the reduction of χ -points as noise in an acyclic sensor network. We introduce a general idea using the notions of pushforwards and pullbacks of functions.

2. DISCRETE EULER CALCULUS FOR FUNCTIONS ON POSETS

We begin by recalling the fundamental notion of discrete Euler calculus. The integration theory with respect to Euler characteristic was originally introduced independently by Viro [Vir88] and Schapira [Sch89]. Subsequently, Baryshnikov and Ghrist proposed its application to sensor networks [BG09]. They established a way to use Euler calculus to enumerate targets lying on a field.

In this paper, we discuss a combinatorial analog of the approach introduced in [Tan16]. A *network* is a finite graph consisting of nodes and lines spanning them. We assume that our network transmits energy or information, or objects. A network is considered *acyclic* if it flows only in one direction (never returning to the original position). Examples of acyclic networks include a stream of a river, the transmission of electricity, and one-way traffic. Acyclic networks such as these can be regarded as a finite poset. Two nodes x and y are ordered $x \leq y$ if the network flows from x to y . Here the Euler calculus is discussed over a function on a finite poset.

The Euler characteristic is well known as a classical topological (homotopical) invariant. It is defined not only for geometric objects, but also for combinatorial objects such as posets. The Euler characteristic of a poset was introduced by Rota [Rot64] using the Möbius inversion. We also refer the readers to Leinster's paper [Lei08] about the Euler characteristic for categories as a generalization of posets.

Definition 2.1. Suppose that P is a finite poset. The *zeta function* of P is a $(0, 1)$ -matrix $\zeta : P \times P \rightarrow \mathbb{Q}$ defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. We can take it as a triangular matrix; hence, this is regular. The *Euler characteristic* $\chi(P)$ of P is defined as the sum of all elements in the inverse matrix ζ^{-1} :

$$\chi(P) = \sum_{x, y} \zeta^{-1}(x, y)$$

This is closely related to the topological Euler characteristic through the order complexes. The *order complex* $\mathcal{K}(P)$ of a poset P is a simplicial complex whose n -simplex is a sequence $p_0 < p_1 < \dots < p_n$. It holds the equality $\chi(P) = \chi(\mathcal{K}(P))$ for any finite poset P . We regard it as a measure on posets, and apply it to the integration of functions on posets.

Definition 2.2. Let P be a finite poset. A *filter* Q is a subposet of P closed under the upper order, i.e., if $x \in Q$ and $y \in P$ with $x \leq y$, then $y \in Q$ whenever. The prime filter $P_{\geq x}$ for a point x in P is defined by $P_{\geq x} = \{y \in P \mid y \geq x\}$. Every filter can be written as a union of some prime filters. Let \mathcal{F}_P denote the collection of filters of P .

We can define the dual notion referred to as an *ideal* above as a subposet closed under the lower order. An ideal of P is a filter of the opposite poset P^{op} ; hence, we only consider filters in this paper. The dual definitions and propositions provided below can be considered for ideals.

Remark 2.3. A finite poset P can be regarded as a finite space with the topology consisting of filters (ideals). Indeed, the category of finite posets and the category of finite T_0 -spaces are equivalent to each other [Sto66], [Bar11]; hence, we identify both in this paper.

As we mentioned earlier, $\chi(P) = \chi(\mathcal{K}(P))$ for a finite poset P . McCord presented a weak homotopy equivalence $K(P) \rightarrow P$ in [McC66]. The topological Euler characteristic depending on the homology groups is an invariant with respect to weak homotopy equivalence; therefore, these Euler characteristics (defined as the alternating sum of the rank of the homology groups) are equal. It states that the combinatorial Euler characteristic for a finite poset P coincides with the topological one for P as a finite space.

An important property of the Euler characteristic with respect to filters is the inclusion-exclusion formula. If Q_1 and Q_2 are filters of a finite poset P , then the following equality holds (Corollary 3.4 in [Tan]):

$$\chi(Q_1 \cup Q_2) = \chi(Q_1) + \chi(Q_2) - \chi(Q_1 \cap Q_2).$$

Using this property, we can establish the integration theory with respect to the Euler characteristic.

Definition 2.4. Let P be a finite poset, and let Q be a subposet of P . The *incidence function* $\delta_Q : P \rightarrow \mathbb{Z}$ is defined by $\delta_Q(x) = 1$ if $x \in Q$, $\delta_Q(x) = 0$ otherwise.

Note that any function $f : P \rightarrow \mathbb{Z}$ on a finite poset P can be written (not uniquely) as a finite linear form $f = \sum_i a_i \delta_{Q_i}$, where $a_i \in \mathbb{Z}$ and $Q_i \in \mathcal{F}_P$. We refer to it as a *filter linear form* of f .

Definition 2.5. Let $f : P \rightarrow \mathbb{Z}$ be a function on a finite poset P with a filter linear form $\sum_i a_i \delta_{Q_i}$. The *Euler calculus* of f is defined as follows:

$$\int_P f d\chi = \sum_i a_i \chi(Q_i).$$

Note that this does not depend on the choice of filter linear forms of f by the inclusion-exclusion formula.

Now we recall our setting of sensor networks for enumerating targets by Euler calculus. We refer the readers to the previous paper [Tan16] for the details. For an acyclic network P , assume that finite targets lie on the network. These are regarded as a discrete subset T in the Hasse diagram (one-dimensional simplicial complex) of P . Examples include line breakage points, bugs or errors, and traffic jams.

Each node is equipped with a sensor detecting targets lying on a lower position than itself. In other words, each sensor at a node $x \in P$ can count the number of targets in the prime ideal $P_{\leq x} = \{y \in P \mid y \leq x\}$. It returns the *counting function* h on P given by $h(x)$ as the number of targets detected by the sensor at x . An *acyclic sensor network* consists of a triple (P, T, h) : the underlying finite poset P , the set of targets T lying on P , and the counting function h on P obtained by the sensors detecting the targets.

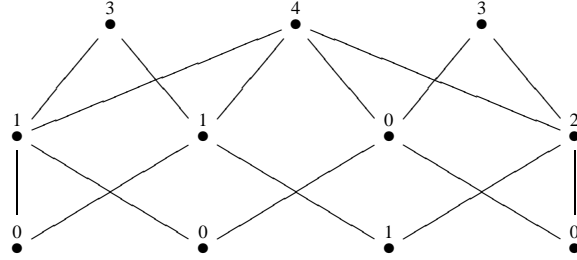
Our main result in [Tan16] was to show that the number of targets can be computed from the Euler calculus of the counting function. This is a combinatorial analog of the work of Baryshnikov and Ghrist [BG09].

Theorem 2.6. *Let (P, T, h) be an acyclic sensor network. The number of targets $T^\#$ is equal to the Euler calculus of the counting function:*

$$\int_P h d\chi = T^\#.$$

Let us consider the following example.

Example 2.7. A counting function h on an acyclic sensor network is described on the following Hasse diagram of a poset P :



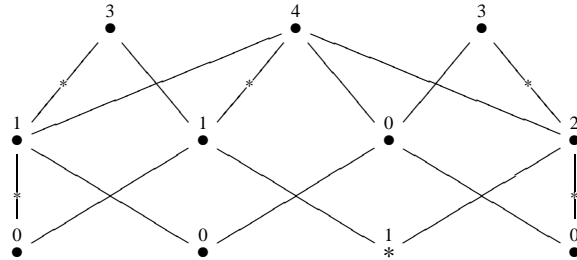
The question is how many targets lie on the network. Theorem 2.6 states that the answer can be found using Euler calculus. Now the counting function h has the following filter linear form using *excursion sets*:

$$h = \delta_{h \geq 1} + \delta_{h \geq 2} + \delta_{h \geq 3} + \delta_{h \geq 4},$$

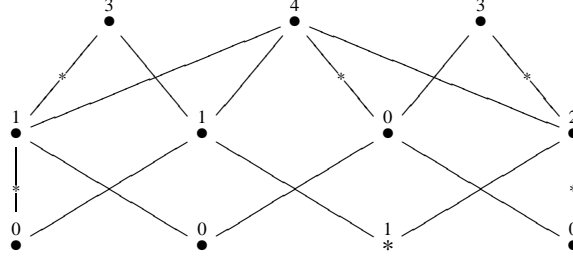
where the excursion set $h \geq i$ is defined as $\{x \in P \mid h(x) \geq i\}$ for each $i \geq 1$. Note that the order complex $h \geq 1$ is homotopy equivalent to a circle S^1 ; hence, the Euler characteristic $\chi(h \geq 1) = \chi(S^1) = 0$. The Euler calculus of h is computed as follows:

$$\int_P h d\chi = \chi(h \geq 1) + \chi(h \geq 2) + \chi(h \geq 3) + \chi(h \geq 4) = 0 + 2 + 3 + 1 = 6.$$

These are six targets lying on the network. Indeed, this counting function is given as the following targets described as the symbol $*$.



Note that we do not know where the targets are. The next diagram arranges the targets such that the positions they occupy differ from those given above, without changing the counting function.



Next, we consider a very simple case. Let (P, T, h) be an acyclic sensor network, and let P have a maximal point x . The sensor at x can detect all targets; hence, $h(x) = T^\# = \int_P h d\chi$ holds. This is a general property of Euler calculus.

Proposition 2.8 (Proposition 3.11 of [Tan16]). *If a finite poset P has a maximal point x , then the Euler calculus of a function h on P is equal to $h(x)$:*

$$\int_P h d\chi = h(x).$$

It states that if P has a maximal point, the other points do not affect the Euler calculus. The maximal point is only essential, and the other points are redundant to enumerate targets in an acyclic sensor network. In section 4, we characterize this noise and remove it in an acyclic network.

3. BEAT POINTS AND WEAK BEAT POINTS

The notions of *beat points* and *weak beat points* play an important role in the homotopy theory of finite space (posets). These points are reducible in the sense of the homotopy theory. A finite poset can be regarded as a finite T_0 -space [Sto66], [Bar11]. Hence, in this paper, we identify both.

Definition 3.1. Let P be a poset. An element x in P is termed a *down-beat point* if the subposet $P_{>x} = \{y \in P \mid y > x\}$ possesses a unique minimal element. *Up-beat points* are defined dually. We refer to a point as simply a *beat point* if it is either a down-beat point or an up-beat point.

A beat point of a poset does not affect the homotopy type of the original poset. We obtain a minimal model with respect to the homotopy type of a finite poset by removing all beat points one by one. This is known as the *core*. It is well known that the core is determined uniquely up to isomorphism, regardless of the order in which the points are removed. Stong classified the homotopy type of finite posets using their cores.

Theorem 3.2 (Theorem 4 of [Sto66]). *Let P and Q be finite posets. P is homotopy equivalent to Q if and only if their cores are isomorphic to each other. In particular, a finite poset P is contractible if and only if the core consists of a single point.*

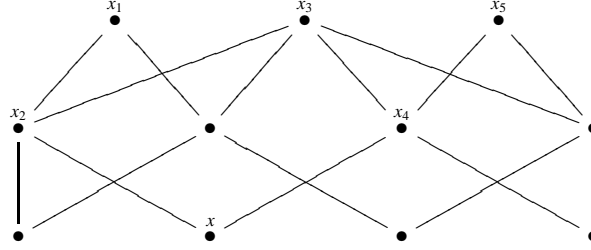
Weak beat points are a generalization of beat points.

Definition 3.3. Let P be a poset. A point x in P is termed a *weak down-beat point* if the subposet $P_{>x} = \{y \in P \mid y > x\}$ is contractible. *Weak up-beat points* are defined dually. We

refer to a point as simply a *weak beat point* if it is either a weak down-beat point or a weak up-beat point.

A weak beat point of a poset does not affect the weak homotopy type or the homotopy type of the order complex. The following example is used to explain the notions of beat points and weak beat points (the opposite poset of Example 4.2.1 of [Bar11]).

Example 3.4. Consider the poset described as the following Hasse diagram.



This poset does not have beat points; however, the point x is a weak down-beat point. Indeed, the subposet $P_{>x}$ can be written as $x_1 > x_2 < x_3 > x_4 < x_5$, and is contractible. By removing the point x from the diagram, the subposet does have beat points and is contractible. Consequently, the original poset is not contractible, but weakly contractible (the order complex is contractible).

We introduce a more general idea of weak down-beat points.

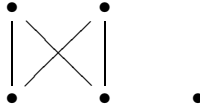
Definition 3.5. Let P be a finite poset. A point x is called a χ -point if $\chi(P_{>x}) = 1$.

Obviously, a weak down-beat point x is a χ -point because $P_{>x}$ is contractible.

Down-beat points \implies Weak down-beat points $\implies \chi$ -points

Unlike (weak) beat points, χ -points are not compatible with the homotopy type. The following example describes a χ -point that is not a weak down-beat point.

Example 3.6. The classifying space of the following poset P is isomorphic to the coproduct of a circle and a single point $S^1 \coprod \{*\}$.



This poset is not contractible; however, the Euler characteristic is $\chi(P) = \chi(\mathcal{K}(P)) = 1$. Consider the poset $\widehat{P} = P \cup \{x\}$ adding a minimal point x to P . The minimal point is a χ -point, but is not a weak down-beat point of \widehat{P} .

For a finite poset P , a χ -minimal model P_χ is a subposet of P formed by removing all χ -points one by one. It is not clear whether this is uniquely determined such as the case of beat points and cores. However, we can obtain the canonical model if we choose a total order on P ; we repeat finding and removing a χ -point in the ascending order.

4. NOISE REDUCTION ON ACYCLIC SENSOR NETWORKS

In Section 2, we have seen that the Euler calculus enumerates targets lying on an acyclic network with sensors. This section discusses the reliability or optimization for this method as a practical problem.

Proposition 4.1. *Let P be a finite poset and let x be a χ -point of P . If two functions h, h' on P satisfies $h(y) = h'(y)$ for any $y \neq x$, then*

$$\int_P h d\chi = \int_P h' d\chi.$$

Proof. Let h_x denote the function on P given by $h_x(x) = 0$ and $h_x(y) = h(y)$ for $x \neq y$. We can write

$$h_x = h - h(x)\delta_{P_{\geq x}} + h(x)\delta_{P_{> x}}.$$

Both $P_{\geq x}$ and $P_{> x}$ have the Euler characteristic 1 by the definition of χ -points.

$$\int_P (h_x) d\chi = \int_P h d\chi - h(x)\chi(P_{\geq x}) + h(x)\chi(P_{> x}) = \int_P h d\chi.$$

The desired result immediately follows from the equality above. \square

Corollary 4.2. *Let h, h' be two functions on a finite poset P . If $h|_{P_\chi} = h'|_{P_\chi}$ for a χ -minimal model P_χ , then*

$$\int_P h d\chi = \int_P h' d\chi.$$

This corollary states that even if the counting function on an acyclic sensor network returns wrong values on χ -points, we can enumerate the correct number of targets by Euler calculus. We can say that χ -points constitute noise in our acyclic sensor networks. On the other hand, from the viewpoint of cost-cutting, we do not need to place sensors at each node to enumerate targets. It is sufficient to locate sensors on the basis of a χ -minimal model.

Next, we discuss the Euler calculus for a function restricted to a χ -minimal model. We naturally expect to hold the following equality for a function h on P :

$$\int_P h d\chi = \int_{P_\chi} h|_{P_\chi} d\chi,$$

for the function restricted to P_χ . Of course, the right-hand side is easy to calculate because the poset is smaller than the left-hand side. We consider a more general setting using pushforwards and pullbacks.

Definition 4.3. Let $f : P \rightarrow Q$ be an order-preserving map between finite poset P and Q . For a function h on P , the *pushforward* f_*h of h along f is a function on Q defined as follows:

$$f_*h(x) = \int_{f^{-1}(P_{\leq x})} h d\chi.$$

The Euler calculus of the pushforward is equal to that of the original function.

Proposition 4.4 (Theorem 3.19 in [Tan16]). *Let $f : P \rightarrow Q$ be an order-preserving map between finite poset P and Q . For a function h on P ,*

$$\int_P h d\chi = \int_Q (f_*h) d\chi.$$

A down-beat point x of a finite poset P determines a retraction $P \rightarrow P \setminus \{x\}$ sending x to the maximal point of $P_{> x}$. The property of this map is characterized as the notion of (*ascending*) *closure operators* (see Section 13.2 in [Koz08]).

Definition 4.5. An order-preserving map $r : P \rightarrow P$ on a poset P is termed an *ascending closure operator* if $r^2 = r$ and $r(x) \geq x$ for any $x \in P$.

Obviously, the composition of the retraction $P \rightarrow P \setminus \{x\}$ given by a down-beat point x of P and the inclusion $P \setminus \{x\} \hookrightarrow P$ is an ascending closure operator. In terms of the homotopy theory for posets, a closure operator $r : P \rightarrow P$ is a deformation retraction onto its image; thus, P and $r(P)$ are homotopy equivalent to each other. We regard a closure operator as a map onto its image $r : P \rightarrow r(P)$, and examine its pushforward.

Proposition 4.6. *If $r : P \rightarrow r(P)$ is an ascending closure operator on a finite poset P , then the pushforward r_*h coincides with the restriction $h|_{r(P)}$ for any function h on P .*

Proof. For a point $x \in r(P)$, there exists $y \in P$ with $r(y) = x$. Consider the filter below:

$$r^{-1}(r(P)_{\leq x}) = \{z \in P \mid r(z) \leq x\}.$$

The point x belongs to this filter as the maximal point since $r(x) = r^2(y) = r(y) = x$, and $z \leq r(z) \leq x$ for any z . Proposition 4.4 and 2.8 lead to our desired formula:

$$r_*h(x) = \int_{r^{-1}(r(P)_{\leq x})} h d\chi = h(x).$$

□

The proposition above and Proposition 4.4 state that we can ignore down-beat points of an acyclic sensor network to enumerate targets. However, does this also apply to weak down-beat points or more general χ -points? In general, these points do not induce a map $P \rightarrow P \setminus \{x\}$ unlike a down-beat point. Now we propose the dual idea of pushforward.

Definition 4.7. For a map $f : P \rightarrow Q$ (not necessarily order preserving) between finite posets P and Q and a function h on Q , the *pullback* f^*h is a function on P defined by the composition $h \circ f$.

Unfortunately, the pullback does not generally hold a similar formula to Proposition 4.4. We find a class of maps on posets compatible with respect to Euler calculus. The following notion of *distinguished maps* was introduced in Definition 4.2 of [BM08].

Definition 4.8. An order preserving map $f : P \rightarrow Q$ is *distinguished* if $f^{-1}(Q_{\geq x})$ is contractible in P for any $x \in Q$.

Quillen's theorem A [Qui73], [McC66] states that a distinguished map is a weak homotopy equivalence on finite posets (or induces a homotopy equivalence on the order complexes). We introduce a more general notion as follows.

Definition 4.9. An order-preserving map $f : P \rightarrow Q$ is χ -*distinguished* if $f^{-1}(Q_{\geq x})$ has the Euler characteristic 1 for any $x \in Q$.

This map is compatible with the pullback and the Euler calculus.

Theorem 4.10. *If $f : P \rightarrow Q$ is a χ -distinguished map, then*

$$\int_Q h d\chi = \int_P (f^*h) d\chi,$$

for a function h on Q .

Proof. For a function h on Q , it can be written as the following filter linear form using prime filters:

$$h = \sum_{x \in Q} a_x \delta_{Q_{\geq x}}.$$

The pullback f^*h maps

$$f^*h(y) = h(f(y)) = \sum_{x \leq f(y)} a_x,$$

for $y \in P$. It implies that f^*h has the following filter linear form:

$$f^*h = \sum_{x \in Q} a_x \delta_{f^{-1}(Q_{\geq x})}.$$

By the definition of χ -distinguished maps, $\chi(f^{-1}(Q_{\geq x})) = 1$ for each $x \in Q$. It shows the desired formula:

$$\int_P (f^*h) d\chi = \sum_{x \in Q} a_x = \int_Q h d\chi.$$

□

For a χ -point x in a finite poset P , the inclusion $P \setminus \{x\} \hookrightarrow P$ is χ -distinguished. The pullback of a function on P along the inclusion is the restriction on $P \setminus \{x\}$.

Theorem 4.11. *Let (P, T, h) be an acyclic sensor network with a χ -minimal model P_χ . If we denote \tilde{P}_χ as a subposet $\{x \in P_\chi \mid h(x) \geq 1\}$, then*

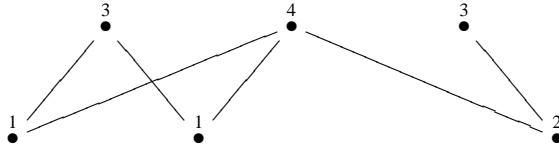
$$\int_{\tilde{P}_\chi} h|_{\tilde{P}_\chi} d\chi = \int_P h d\chi.$$

Proof. Theorem 4.10 leads to the following equality:

$$\int_{P_\chi} h|_{P_\chi} d\chi = \int_P h d\chi.$$

As we have seen in Example 2.7, the Euler calculus of the counting function can be computed as the telescope sum of the Euler characteristics for excursion sets. The points taking the value zero do not affect the result. □

Recall the counting function in Example 2.7. We can remove the two middle-bottom weak down-beat points, and the points taking the value zero as follows:



The Euler calculus of this function returns the number of targets lying on the original network.

As another example, in the last part of Section 2 we mentioned the case in which the underlying poset P has a maximal point x . In this case, any point except for the maximal point x is a weak down-beat point and reducible. We can show the formula in Proposition 2.8 from Theorem 4.11:

$$\int_P h d\chi = \int_{\{x\}} h|_{\{x\}} d\chi = h(x).$$

Concluding remarks and future work. The research of this paper has established a way to simplify the calculation to enumerate targets lying on an acyclic network. These methods with respect to discrete Euler calculus are suitable for computer calculations such as finding χ -points and removing them, taking excursion sets, and computing (discrete) Euler characteristics. It is advantageous to calculate discrete Euler calculus using a computer.

This research is based on acyclic networks, however, we can consider more complex networks, such as having multiple lines or cycles. Our next step is to establish a way to enumerate targets lying on such networks. In these cases, it may help to use categories instead of posets. As another remaining issue, it is important to guarantee the universality of χ -minimal model for building a computer program.

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